Math 262 **Test #2 Solutions** **Spring 2005**

1. Write
$$\sum_{n=1}^{5} \frac{(-1)^{n+1}2^{n-1}}{n+1}.$$
This could be simplified to
$$\sum_{n=1}^{5} \frac{(-2)^{n-1}}{n+1},$$
or
$$\sum_{n=0}^{4} \frac{(-2)^{n}}{n+2}.$$

2. The simplest method is to use algebra:
$$\begin{aligned}\sum_{k=1}^{n}(2k+3) &= 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3 \\ &= 2\left(\frac{n(n+1)}{2}\right) + 3n \\ &= n(n+1) + 3n \\ &= n^2 + 4n.\end{aligned}$$
Another method would be to find a formula for the sum of the odd integers (we did that in class), namely,
$$\sum_{k=1}^{n}(2k-1) = n^2.$$
Then apply this to the given sum:
$$\begin{aligned}\sum_{k=1}^{n}(2k+3) &= \sum_{k=3}^{n+2}(2k-3) \\ &= \sum_{k=1}^{n+2}(2k-3) - (1+3) \\ &= (n+2)^2 - 4 \\ &= n^2 + 4n.\end{aligned}$$

3. When $n = 1$, $\sum_{k=1}^{n} k(k+1) = 1 \cdot 2$ and $\frac{n(n+1)(n+2)}{3} = \frac{1 \cdot 2 \cdot 3}{3} = 2$. Therefore, the statement is true when $n = 1$.

Now suppose that the statement is true when $n = m$, for some integer $m \geq 1$. We will show that it is true when $n = m + 1$.

$$\begin{aligned}
\sum_{k=1}^{m+1} k(k+1) &= \sum_{k=1}^{m} k(k+1) + (m+1)(m+2) \\
&= \frac{m(m+1)(m+2)}{3} + \frac{3(m+1)(m+2)}{3} \\
&= \frac{(m+1)(m+2)(m+3)}{3}
\end{aligned}$$

Therefore, the statement is true for all $n \geq 1$.

4. Let $n = 1$. Then $a_n = a_1 = 1$ and $3 \cdot 2^n - 5 = 3 \cdot 2^1 - 5 = 1$. Therefore, the statement is true when $n = 1$.

   Now suppose that the statement is true when $n = k$, for some integer $k \geq 1$. We will show that the statement is true when $n = k + 1$.

$$\begin{aligned}
a_{n+1} &= 2a_n + 5 \\
&= 2\,(3 \cdot 2^n - 5) + 5 \\
&= \left(3 \cdot 2^{n+1} - 10\right) + 5 \\
&= 3 \cdot 2^{n+1} - 5.
\end{aligned}$$

   Therefore, the statement is true for all $n \geq 1$.

5. Let $A = \{1, 2, 4, 8\}$, $B = \{1, 3, 5, 7\}$, and $C = \{2, 3, 5, 7\}$ and let the universal set be $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Then

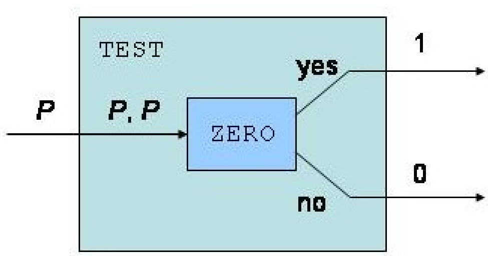
   (a) $A \cup B = \{1, 2, 3, 4, 5, 7, 8\}$

   (b) $B \cap C = \{3, 5, 7\}$

   (c) $(A \cup B) - (A \cap B) = \{2, 3, 4, 5, 7, 8\}$

   (d) $(B \cap C)^c = \{1, 2, 4, 6, 8\}$

6. Suppose such a program `ZERO` exists. Then use it to build a program `TEST` as follows:



   The program `TEST` will read a program P. It will then run `ZERO` with input P (as the program) and P (as the input to P). `ZERO` will determine whether P

will output 0 on input P. If `ZERO` reports "yes," then `TEST` will output 1, and if `ZERO` reports "no," then `TEST` will output 0. Now run `TEST` on input `TEST` and we will have a contradiction. If `TEST` should output 0 on input `TEST`, then `ZERO` will report "yes," so `TEST` will output 1. And if `TEST` should not output 0 on input `TEST`, then `ZERO` will report "no," so `TEST` will output 0.

7. (a) There are six possible choices:

$$\{R_1R_2, R_1B_1, R_1B_2, R_2B_1, R_2B_2, B_1B_2\}.$$

   (b) Of the six equally likely choices in part (a), 2 of them are of the same color $\{R_1R_2, B_1B_2\}$. Therefore, the probability is $\frac{2}{6} = \frac{1}{3}$.

8. (a) There are $\left\lfloor \frac{10000}{29} \right\rfloor = 344$ multiples of 29, $\left\lfloor \frac{10000}{37} \right\rfloor = 270$ multiples of 37, and $\left\lfloor \frac{10000}{89} \right\rfloor = 112$ multiples of 89. Furthermore, there are $\left\lfloor \frac{10000}{29\cdot 37} \right\rfloor = 9$ multiples of $29 \cdot 37$, $\left\lfloor \frac{10000}{29\cdot 89} \right\rfloor = 3$ multiples of $29 \cdot 89$, and $\left\lfloor \frac{10000}{37\cdot 89} \right\rfloor = 3$ multiples of $37 \cdot 89$. There are $\left\lfloor \frac{10000}{29\cdot 37\cdot 89} \right\rfloor = 0$ multiples of $29 \cdot 37 \cdot 89$. Using the Inclusion-Exclusion Principle, the number of numbers from 1 to 10000 that are multiples of 29, 37, or 89 is

$$344 + 270 + 112 - 9 - 3 - 3 + 0 = 711.$$

   (b) If 711 of the numbers are multiples of 29, 37, or 89, then $10000 - 711 = 9289$ of the numbers are not multiples of 29, 37, or 89. If we choose one at random, the probability is $\frac{9289}{10000} = 0.9289$ that it is not a multiple of 29, 37, or 89.

9. The total number of ways to choose any 3 marbles from the 10 is $\binom{10}{3} = 120$. To choose 3 marbles of the same color is to choose 3 red marbles and 0 green marbles or 0 red marbles and 3 green marbles. The number of ways to do this is $\binom{5}{3}\binom{5}{0} + \binom{5}{0}\binom{5}{3} = 10 \cdot 1 + 1 \cdot 10 = 20$. So the probability is $\frac{20}{120} = \frac{1}{6}$.

10. Expand as

$$\begin{aligned}(a+2b)^5 &= a^5 + \binom{5}{1}a^4(2b) + \binom{5}{2}a^3(2b)^2 + \binom{5}{3}a^2(2b)^3 + \binom{5}{4}a(2b)^4 + (2b)^5 \\ &= a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5.\end{aligned}$$